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A Condition for Finite Blow-up Time for a Volterra Integral Equation

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The paper is devoted to the study of the equation

$$u(x) = \int_0^x (x-s)^{\alpha-1}\, g(u(s))\, ds \qquad (\alpha > 0,\ x > 0),$$

where g is continuous and nondecreasing. Some estimates of u' and the interval of the existence of a nontrivial solution u are given. © 1994 Academic Press, Inc.

1. Introduction

In this paper we consider the following integral equation

$$u(x) = \int_0^x (x-s)^{\alpha-1}\, g(u(s))\, ds \qquad (x > 0,\ \alpha > 0), \tag{1.1}$$

where

(i) g is continuous and nondecreasing with $g(0) = 0$ and $g(x) > 0$ for $x > 0$;

(ii) g satisfies the following generalized Osgood condition ([1, 2, 4]),

$$\int_0^\delta \left(\frac{u}{g(u)}\right)^{1/\alpha} \frac{du}{u} < \infty \qquad (\delta > 0).$$

We are interested only in continuous, nonnegative solutions of (1.1). It is clear that $u \equiv 0$ is a solution of (1.1). In [4] it is shown that under the assumptions i and ii there are other, nontrivial, solutions. Moreover, from classical theorems on the existence of the maximal solution for Volterra integral equations (see [3, Chap. II]) and ideas of [1, 5], we have

Remark 1.1. If i and ii are satisfied, then there exists a maximal solu-

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tion u of (1.1). It is a nondecreasing function and a unique solution with the property $u(x) > 0$ for $x > 0$.

From now on we deal with a maximal solution u of (1.1). Since it is nondecreasing, we get

$$u(x) = 1/\alpha \int_0^x (x-s)^\alpha \, dg(u(s)) \qquad (x > 0), \tag{1.2}$$

from which follows

Remark 1.2. u is an absolutely continuous function.

Our aim is to estimate $(u^{-1})'$, where u^{-1} is the inverse to u, from which we obtain estimates of the interval $[0, \bar{K})$ of the existence of u. The main results are stated in the following theorem and its corollary.

THEOREM 1.1. *There exist* $\gamma_1, \gamma_2 > 0$ *such that*

$$\gamma_1 u'(x) \leqslant \left\{ \int_0^x [u(x) - u(s)]^{\alpha - 1} \, dg(u(s)) \right\}^{1/\alpha} \leqslant \gamma_2 u'(x),$$

for $0 \leqslant x \leqslant \bar{K}$.

If

$$\psi(x) = \left\{ \int_0^x [u(x) - u(s)]^{\alpha - 1} \, dg(u(s)) \right\}^{1/\alpha} \qquad (0 < x \leqslant \bar{K})$$

and $\phi(y) = \psi(u^{-1}(y))$, then from Theorem 1.1 we obtain immediately

COROLLARY 1.1. (i) $\gamma_1 \phi(y) \leqslant (u^{-1})'(y) \leqslant \gamma_2 \phi(y)$;

(ii) $\gamma_1 \int_0^\infty \phi(y) \, dy \leqslant \bar{K} \leqslant \gamma_2 \int_0^\infty \phi(y) \, dy$.

Hence, $\bar{K}$ is finite if and only if $\int_0^\infty \phi(y) \, dy < \infty$, where $\phi(y) = \{\int_0^y (y-s)^{\alpha-1} \, dg(s)\}^{-1/\alpha}$. The last statement leads to the following theorem.

THEOREM 1.2. *The interval* $[0, \bar{K})$ *of the existence of* u *is finite if and only if*

$$\int_0^\infty \left[\frac{s}{g(s)} \right]^{1/\alpha} \frac{ds}{s} < \infty.$$

2. PROOF OF THE THEOREMS

To prove Theorem 1.1 we need two lemmas, in which we consider the cases $0 < \alpha \leqslant 1$ and $\alpha > 1$, respectively.

Let

$$\Phi(s)=\int_0^s (x-t)^{\alpha-1}\, dg(u(t)) \qquad \text{for} \quad 0\leqslant s\leqslant x.$$

LEMMA 2.1. *If* $0<\alpha\leqslant 1$, *then for* $0\leqslant x<\bar{K}$

$$(x-y)\,\Phi(y)\leqslant u(x)-u(y)\leqslant (1/\alpha)(x-y)\,u'(x). \tag{2.1}$$

Proof. By the Mean Value Theorem we have

$$\alpha(x-s)^{\alpha-1}(x-y)\leqslant (x-s)^{\alpha}-(y-s)^{\alpha} \qquad \text{for} \quad 0<s<y\leqslant x. \tag{2.2}$$

Since, by $0<\alpha\leqslant 1$, $(y-s)^{\alpha}\geqslant (x-s)^{\alpha-1}(y-s)$ for $0<s<y\leqslant x$, we get

$$(x-s)^{\alpha}-(y-s)^{\alpha}\leqslant (x-s)^{\alpha-1}(x-y) \qquad \text{for} \quad 0<s<y\leqslant x. \tag{2.3}$$

Note also that by (1.2)

$$\alpha(u(x)-u(y))=\int_0^y \{(x-s)^{\alpha}-(y-s)^{\alpha}\}\, dg(u(s))+\int_y^x (x-s)^{\alpha}\, dg(u(s)).$$

Hence using (2.2) and (2.3) we can obtain the required assertion.

COROLLARY 2.1. *If* $0<\alpha\leqslant 1$, *then for* $0\leqslant x<\bar{K}$

$$\alpha^{1-\alpha}[u'(x)]^{\alpha}\leqslant [\psi(x)]^{-\alpha}\leqslant \alpha^{-1}[u'(x)]^{\alpha}. \tag{2.4}$$

Proof. Since $\alpha-1\leqslant 0$, from the right-hand side inequality of (2.1) it follows that

$$[\psi(x)]^{-\alpha}=\int_0^x [u(x)-u(y)]^{\alpha-1}\, dg(u(y))$$

$$\geqslant \alpha^{-(\alpha-1)}[u'(x)]^{\alpha-1}\int_0^x (x-y)^{\alpha-1}\, dg(u(y))=\alpha^{1-\alpha}[u'(x)]^{\alpha}.$$

Noting that $d\Phi(y)=(x-y)^{\alpha-1}\, dg(u(y))$ and that $\Phi(x)=u'(x)$, and using the left-hand side inequality of (2.1), we obtain

$$[\psi(x)]^{-\alpha}\leqslant \int_0^x (x-y)^{\alpha-1}\, \Phi(y)^{\alpha-1}\, dg(u(y))$$

$$=\int_0^x \Phi(y)^{\alpha-1}\, d\Phi(y)=(1/\alpha)[u'(x)]^{\alpha},$$

which gives the right-hand side inequality of (2.4).

The second lemma concerns the case $\alpha > 1$. The right-hand side inequality is an easy consequence of the Mean Value Theorem. The left-hand side inequality is difficult and depends on an inductive argument.

LEMMA 2.2. *If* $\alpha > 1$, *then for* $0 \leqslant x < \bar{K}$,

$$\alpha^{-\alpha}[u'(x)]^{\alpha} \leqslant [\psi(x)]^{-\alpha} \leqslant [u'(x)]^{\alpha}. \tag{2.5}$$

Proof. We restrict ourself to the left-hand side inequality only.

Let $1 < \alpha = n + \beta$, where $0 < \beta \leqslant 1$ and n is an integer. Obtaining $u^{(k+1)}$ $(0 \leqslant k \leqslant n-1)$ from (1.2) and then using the Schwarz inequality, we get

$$[u^{(k+1)}(x)]^2 \leqslant \frac{\alpha - k}{\alpha - k - 1} u^{(k)}(x)\, u^{(k+2)}(x) \tag{2.6}$$

for $0 \leqslant k \leqslant n-1$ and $0 \leqslant x < \bar{K}$.

The inequality

$$u''(x)\, u^{(k)}(x) \leqslant \frac{\alpha - 1}{\alpha - k} u'(x)\, u^{(k+1)}(x), \qquad 1 \leqslant k \leqslant n, \tag{2.7}$$

for $0 \leqslant x < \bar{K}$ follows from the following inductive arguments. For $k = 1$ (2.7) is obvious. Assume (2.7) for $1 \leqslant k \leqslant n-1$, then for $k+1$ we have

$$\begin{aligned} u''(x)\, u^{(k+1)}(x) &= u''(x)\, u^{(k)}(x)\, u^{(k+1)}(x)[u^{(k)}(x)]^{-1} \\ &\leqslant \frac{\alpha - 1}{\alpha - k} u'(x)[u^{(k+1)}(x)]^2\, [u^{(k)}(x)]^{-1}. \end{aligned}$$

Now, it suffices to apply (2.6) to get the required assertion.

Since for $0 \leqslant k \leqslant n-1$,

$$\begin{aligned} \{[u'(x)]^{k+\beta}\, u^{(n-k)}(x)\}' = {} & (k+\beta)[u'(x)]^{k+\beta-1}\, u''(x)\, u^{(n-k)}(x) \\ & + [u'(x)]^{k+\beta}\, u^{(n+1-k)}(x), \end{aligned}$$

from (2.7) we obtain

$$\{[u'(x)]^{k+\beta}\, u^{(n-k)}(x)\}' \leqslant \alpha[u'(x)]^{k+\beta}\, u^{(n+1-k)}(x) \qquad (0 \leqslant k \leqslant n-1). \tag{2.8}$$

Now we are ready to consider the left-hand side inequality (2.5). We obtain it taking $k = n$ in the inequality

$$\int_0^x \{u(x) - u(y)\}^{k+\beta-1}\, dg(u(y)) \geqslant \alpha^{-k+1} c_{n-k}[u'(x)]^{k+\beta-1}\, u^{(n+1-k)}(x), \tag{2.9}$$

where $c_0 = 1/\alpha$, $c_{n-k} = \{(n+\beta)\cdots(k+\beta)\}^{-1}$ for $0 \leqslant k \leqslant n-1$ and $0 \leqslant x < \bar{K}$, which depends on the following inductive argument.

Let $k=0$. Since $\beta - 1 \leqslant 0$ and u' is nondecreasing, an application of the Mean Value Theorem gives

$$\int_0^x \{u(x)-u(y)\}^{\beta-1}\, dg(u(y)) \geqslant \alpha c_n [u'(x)]^{\beta-1}\, u^{(n+1)}(x).$$

Assume (2.9) for $0 \leqslant k \leqslant n-1$. Then for $k+1$ we obtain

$$\begin{aligned}
&\int_0^x \{u(x)-u(s)\}^{k+\beta}\, dg(u(s)) \\
&\qquad = (k+\beta) \int_0^x u'(s) \int_0^s \{u(s)-u(t)\}^{k+\beta-1}\, dg(u(t))\, ds \\
&\qquad \geqslant \alpha^{-k+1}(k+\beta)\, c_{n-k} \int_0^x [u'(s)]^{k+\beta}\, u^{(n+1-k)}(s)\, ds.
\end{aligned}$$

Since $c_{n-(k+1)} = (k+\beta)\, c_{n-k}$ and $u^{(n-k)}(0) = 0$ for $0 \leqslant k \leqslant n-1$, it suffices to apply the inequality (2.8) to obtain the required assertion.

Proof of Theorem 1.2. First we consider the case $0 < \alpha \leqslant 1$. An application of the Jensen inequality gives

$$\begin{aligned}
&1/a \int_a^{2a} \left\{ \int_0^s (s-t)^{\alpha-1}\, dg(t) \right\}^{-1/\alpha} ds \\
&\qquad \geqslant a^{1/\alpha} \left\{ \int_a^{2a} \int_0^s (s-t)^{\alpha-1}\, dg(t)\, ds \right\}^{-1/\alpha} \qquad (a>0). \qquad (2.10)
\end{aligned}$$

Since

$$\int_0^{2a} \int_0^s (s-t)^{\alpha-1}\, dg(t)\, ds \leqslant 1/\alpha \int_0^{2a} (2a-s)^\alpha\, dg(s) \leqslant (1/\alpha)(2a)^\alpha\, g(2a),$$

from (2.10) we get

$$\begin{aligned}
\int_a^{2a} \left\{ \int_0^s (s-t)^{\alpha-1}\, dg(t) \right\}^{-1/\alpha} ds &\geqslant 2^{-1}(\alpha a)^{1/\alpha}\, [g(2a)]^{-1/\alpha} \\
&\geqslant c \int_{2a}^{4a} \left[\frac{s}{g(s)} \right]^{1/\alpha} \frac{ds}{s},
\end{aligned}$$

where $c = 2^{-1}(4^{1/\alpha} - 2^{1/\alpha})^{-1}\, \alpha^{1/\alpha - 1}$. Now it is a simple matter to get

$$\int_0^\infty \phi(y)\, dy \geqslant c \int_0^\infty \left[\frac{s}{g(s)} \right]^{1/\alpha} \frac{ds}{s}.$$

The estimate

$$\int_0^s (s-t)^{\alpha-1}\, dg(t) \geqslant s^{\alpha-1} g(s)$$

leads to the inequality

$$\int_0^\infty \phi(y)\, dy \leqslant \int_0^\infty \left[\frac{s}{g(s)}\right]^{1/\alpha} \frac{ds}{s},$$

which ends the proof of the theorem in the case of $0<\alpha\leqslant 1$.

In the case $\alpha>1$ the required assertion is a simple consequence of the inequality

$$(s/2)^{\alpha-1}\, g(s/2) \leqslant \int_0^s (s-t)^{\alpha-1}\, dg(t) \leqslant s^{\alpha-1} g(s).$$

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